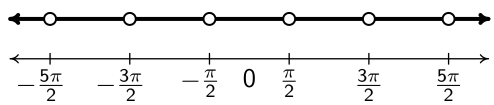
MATH 1700: SECTION 10.5: GRAPHS OF THE OTHER CIRCULAR FUNCTIONS

GRAPHS OF SECANT AND COSECANT:

To get started graphing $F(t) = \sec(t)$, we rewrite $F(t) = \sec(t) = \frac{1}{\cos(t)}$. We see that $F(t)$ is undefined whenever $\cos(t) = 0$, or when $t = \frac{\pi}{2} + \pi k$ for integers k.

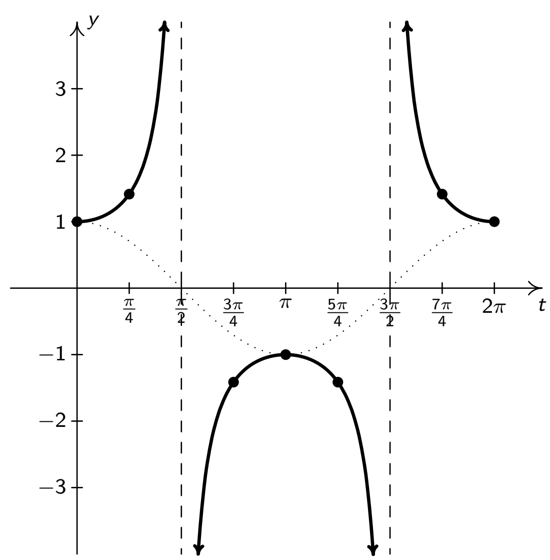
This gives us one way to describe the domain of F: $\{t \mid t \neq \frac{\pi}{2} + \pi k, \text{for integers } k\}$. To get a better feel for the set of real numbers we're dealing with, we write out and graph the domain on the number line.

Running through a few values of k, we find some of the values excluded from the domain: $t \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$. Using these we can graph the domain on the number line below.



Plotting some 'common' values, we note that as we approach values where $\cos(t) = 0$, $\sec(t) \to \pm\infty$, depending on if we are taking the reciprocal of positive or negative values. This produces vertical asymptotes in the graph.

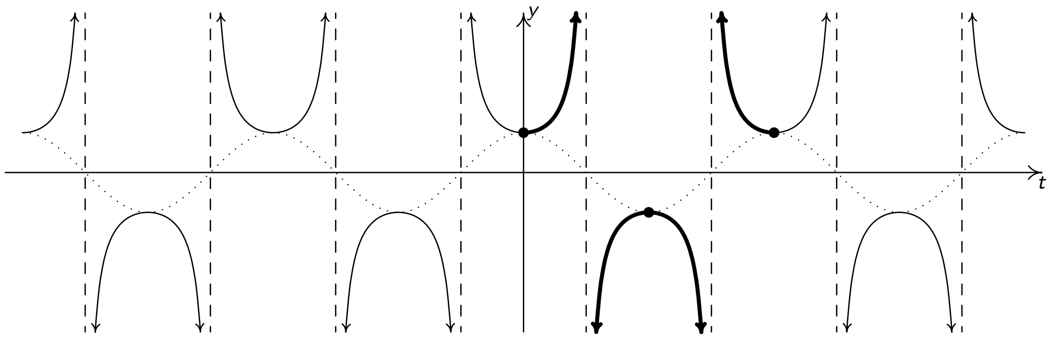
t	$\cos(t)$	$\sec(t)$	$(t, \sec(t))$
0	1	1	$(0,1)$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$	$\left(\frac{\pi}{4}, \sqrt{2}\right)$
$\frac{\pi}{2}$	0	undefined	
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\sqrt{2}$	$\left(\frac{3\pi}{4}, -\sqrt{2}\right)$
π	-1	-1	$(\pi, -1)$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\sqrt{2}$	$\left(\frac{5\pi}{4}, -\sqrt{2}\right)$
$\frac{3\pi}{2}$	0	undefined	
$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$	$\left(\frac{7\pi}{4}, \sqrt{2}\right)$
2π	1	1	$(2\pi, 1)$



The 'fundamental cycle' of $y = \sec(t)$.

To get a graph of the entire secant function, we paste copies of the fundamental cycle end to end to produce the graph. Since $\cos(t)$ is even, that is, $\cos(-t) = \cos(t)$, we have $\sec(-t) = \frac{1}{\cos(-t)} = \frac{1}{\cos(t)} = \sec(t)$.

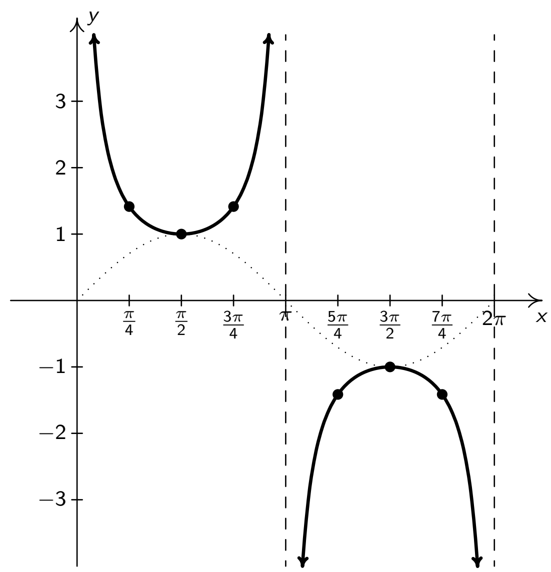
Hence, along with its period, the secant function inherits its symmetry from the cosine function.



The graph of $y = \sec(t)$.

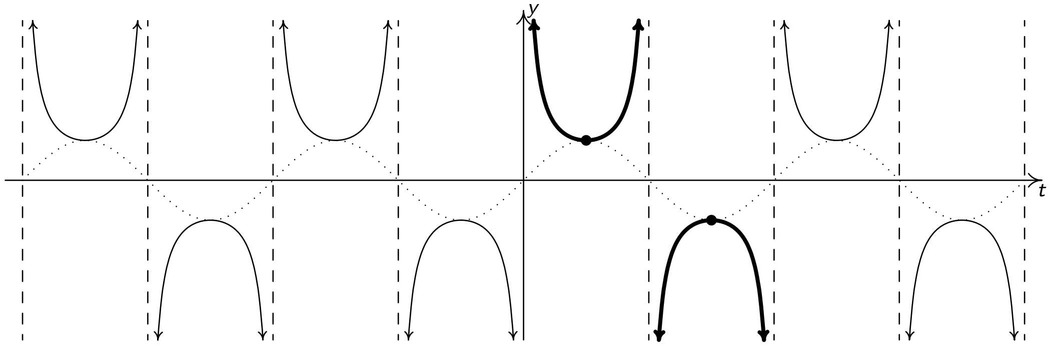
As one would expect, to graph $G(t) = \csc(t)$ we begin with $y = \sin(t)$ and take reciprocals of the corresponding y-values. Here, we encounter issues at $t = 0$, $t = \pi$, $t = 2\pi$, and, in general, at all whole number multiples of π, so the domain of G is $\{t \mid t \neq \pi k, \text{for integers } k\}$. Not surprisingly, these values produce vertical asymptotes.

t	$\sin(t)$	$\csc(t)$	$(t, \csc(t))$
0	0	undefined	
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$	$\left(\frac{\pi}{4}, \sqrt{2}\right)$
$\frac{\pi}{2}$	1	1	$\left(\frac{\pi}{2}, 1\right)$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$	$\left(\frac{3\pi}{4}, \sqrt{2}\right)$
π	0	undefined	
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\sqrt{2}$	$\left(\frac{5\pi}{4}, -\sqrt{2}\right)$
$\frac{3\pi}{2}$	-1	-1	$\left(\frac{3\pi}{2}, -1\right)$
$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\sqrt{2}$	$\left(\frac{7\pi}{4}, -\sqrt{2}\right)$
2π	0	undefined	



The 'fundamental cycle' of $y = \csc(t)$.

We 'copy and paste' to generate the complete graph of $G(t) = \csc(t)$. We note that $G(t) = \csc(t)$ inherits odd symmetry from $\sin(t)$ in the same way $F(t) = \sec(t)$ inherits even symmetry from $\cos(t)$.



The graph of $y = \csc(t)$.

PROPERTIES OF THE SECANT AND COSECANT FUNCTIONS:

- The function $F(t) = \sec(t)$
  - has domain $\left\{t \mid t \neq \frac{\pi}{2} + \pi k,\ k \text{ is an integer}\right\}$
  - has range $(-\infty, -1] \cup [1, \infty)$
  - is continuous and smooth on its domain
  - is even
  - has period 2π
- The function $G(t) = \csc(t)$
  - has domain $\{t \mid t \neq \pi k,\ k \text{ is an integer}\}$
  - has range $(-\infty, -1] \cup [1, \infty)$
  - is continuous and smooth on its domain
  - is odd
  - has period 2π
- Conversion formulas: $\csc\left(t + \frac{\pi}{2}\right) = \sec(t)$ and $\sec\left(t - \frac{\pi}{2}\right) = \csc(t)$

EXAMPLE 1: Graph one cycle of the following functions. State the period of each.

1. $f(t) = 1 - 2\sec(2t)$
2. $g(t) = \dfrac{\csc(-\pi t - \pi) - 5}{3}$

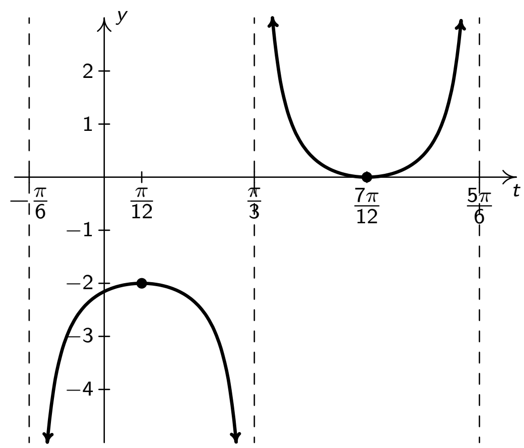
CHARACTERISTICS OF SECANT AND COSECANT CURVES:

For $\omega > 0$, the graphs of

$$F(t) = A\sec(\omega t + \phi) + B \quad \text{and} \quad G(t) = A\csc(\omega t + \phi) + B$$

- have period $T = \dfrac{2\pi}{\omega}$
- have phase shift $-\dfrac{\phi}{\omega}$
- have 'baseline' B and have a vertical gap $|A|$ units between the the baseline and the graph.

EXAMPLE 2: Below is the graph of one cycle of a secant (cosecant) function, $y = f(t)$.



1. Write $f(t)$ in the form $F(t) = A\sec(\omega t + \phi) + B$ for $\omega > 0$.

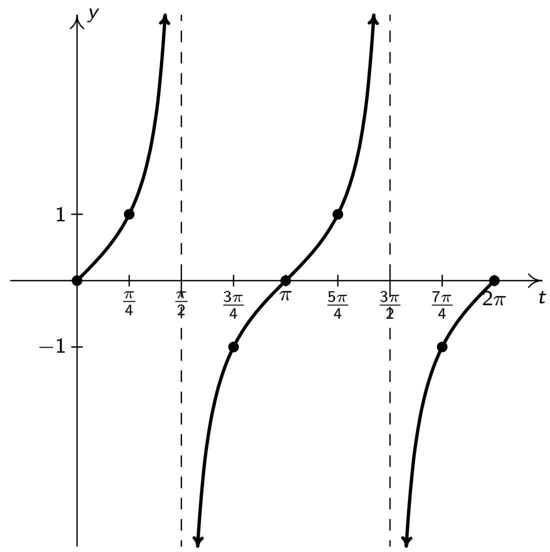
2. Write $f(t)$ in the form $G(t) = A\csc(\omega t + \phi) + B$ for $\omega > 0$.

GRAPHS OF TANGENT AND COTANGENT FUNCTIONS:

Next, we turn our attention to the tangent and cotangent functions. Viewing $J(t) = \tan(t) = \frac{\sin(t)}{\cos(t)}$, we find the domain of J excludes all values where $\cos(t) = 0$. Hence, the domain of J is $\{t \mid t \neq \frac{\pi}{2} + \pi k, \text{for integers } k\}$.

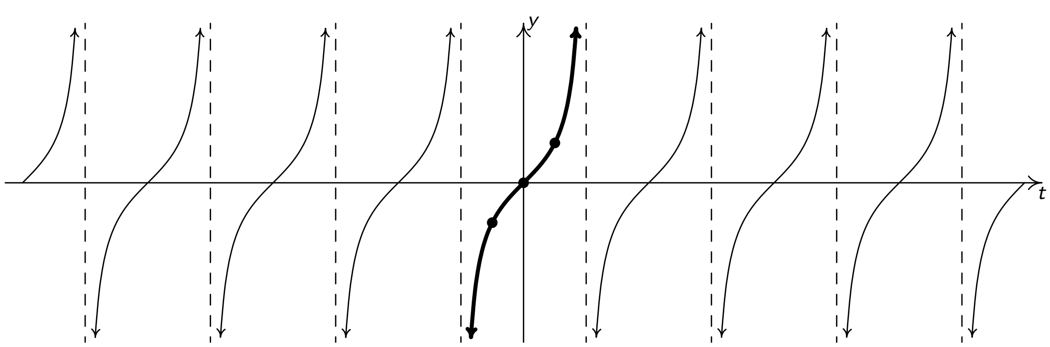
As with the secant function above, we see as we approach values where $\cos(t) = 0$, $\tan(t) \to \pm\infty$.

t	$\tan(t)$	$(t, \tan(t))$
0	0	$(0,0)$
$\frac{\pi}{4}$	1	$\left(\frac{\pi}{4}, 1\right)$
$\frac{\pi}{2}$	undefined	
$\frac{3\pi}{4}$	-1	$\left(\frac{3\pi}{4}, -1\right)$
π	0	$(\pi, 0)$
$\frac{5\pi}{4}$	1	$\left(\frac{5\pi}{4}, 1\right)$
$\frac{3\pi}{2}$	undefined	
$\frac{7\pi}{4}$	-1	$\left(\frac{7\pi}{4}, -1\right)$
2π	0	$(2\pi, 0)$



The graph of $y = \tan(t)$ over $[0, 2\pi]$.

After the usual 'copy and paste' procedure, we create the graph of $y = \tan(t)$ below:



The graph of $y = \tan(t)$.

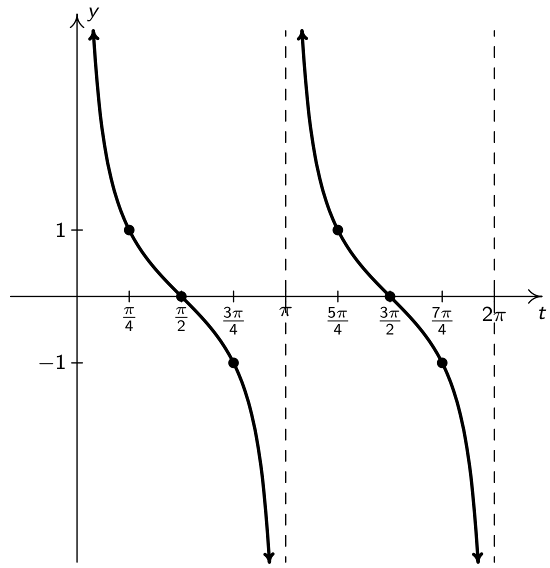
Since sine is odd and cosine is even, we get tangent is odd: $\tan(-t) = \frac{\sin(-t)}{\cos(-t)} = \frac{-\sin(t)}{\cos(t)} = -\tan(t)$.

As we mentioned in the previous section, the period of tangent (and cotangent) is π so we've chosen to select the (open) interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ as highlighted above to represent the 'fundamental cycle' of the tangent function. In addition to the asymptotes at the endpoints $t = \pm\frac{\pi}{2}$, we use the 'quarter marks' $t = \pm\frac{\pi}{4}$ and $t = 0$.

Since $\cot(t) = \frac{\cos(t)}{\sin(t)}$, the domain of $K(t) = \cot(t)$ excludes the values where $\sin(t) = 0$: $\{t \mid t \neq \pi k, \text{for integers } k\}$.

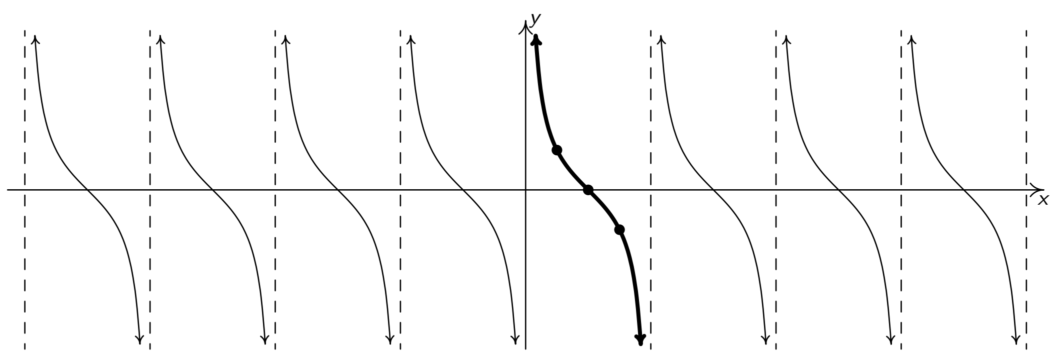
After analyzing the behavior of K near the values excluded from its domain along with plotting points, we graph $y = \cot(t)$ over the interval $[0, 2\pi]$ below on the right.

t	$\cot(t)$	$(t, \cot(t))$
0	undefined	
$\frac{\pi}{4}$	1	$\left(\frac{\pi}{4}, 1\right)$
$\frac{\pi}{2}$	0	$\left(\frac{\pi}{2}, 0\right)$
$\frac{3\pi}{4}$	-1	$\left(\frac{3\pi}{4}, -1\right)$
π	undefined	
$\frac{5\pi}{4}$	1	$\left(\frac{5\pi}{4}, 1\right)$
$\frac{3\pi}{2}$	0	$\left(\frac{3\pi}{2}, 0\right)$
$\frac{7\pi}{4}$	-1	$\left(\frac{7\pi}{4}, -1\right)$
2π	undefined	



The graph of $y = \cot(t)$ over $[0, 2\pi]$.

As usual, pasting copies end to end produces the graph of $K(t) = \cot(t)$ below.



The graph of $y = \cot(t)$.

As with $J(t) = \tan(t)$, the graph of $K(t) = \cot(t)$ is odd and has period π. Also, we see that the period of cotangent (like tangent) is π. We take as one fundamental cycle the graph as traced out over the interval $(0, \pi)$ with quarter marks: $t = 0$, $t = \frac{\pi}{4}$, $t = \frac{\pi}{2}$, $t = \frac{3\pi}{4}$ and $t = \pi$.

PROPERTIES OF THE TANGENT AND COTANGENT FUNCTIONS:

- The function $J(t) = \tan(t)$
  - has domain $\left\{t \,|\, t \neq \frac{\pi}{2} + \pi k,\ k \text{ is an integer}\right\}$
  - has range $(-\infty, \infty)$
  - is continuous and smooth on its domain
  - is odd
  - has period π
- The function $K(t) = \cot(t)$
  - has domain $\{t \,|\, t \neq \pi k,\ k \text{ is an integer}\}$
  - has range $(-\infty, \infty)$
  - is continuous and smooth on its domain
  - is odd
  - has period π
- Conversion formulas: $\tan\left(t + \frac{\pi}{2}\right) = -\cot(t)$ and $\cot\left(t - \frac{\pi}{2}\right) = -\tan(t)$

EXAMPLE 3: Graph one cycle of the following functions. Find the period.

1. $f(t) = 1 - \tan\left(\frac{t}{2} - \pi\right)$.
2. $g(t) = 2\cot(2\pi - \pi t) - 1$.

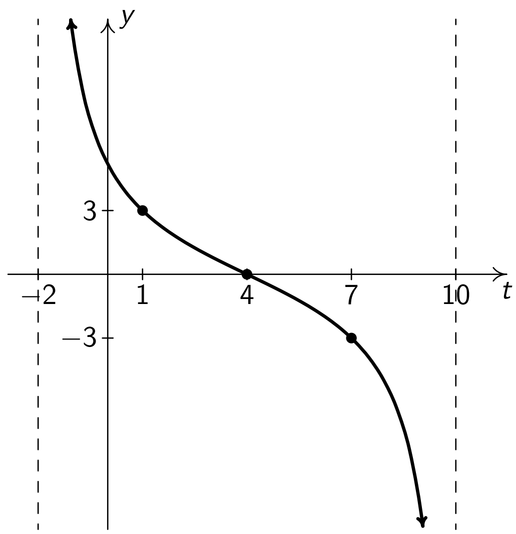
CHARACTERISTICS OF TANGENT AND COTANGENT CURVES:

For $\omega > 0$, the functions

$$J(t) = A\tan(\omega t + \phi) + B \quad \text{and} \quad K(t) = A\cot(\omega t + \phi) + B$$

- have period $T = \dfrac{\pi}{\omega}$
- have vertical shift or 'baseline' B
- The phase shift for $y = J(t)$ is $-\dfrac{\phi}{\omega} - \dfrac{\pi}{2\omega}$.
- The phase shift for $y = K(t)$ is $-\dfrac{\phi}{\omega}$.

EXAMPLE 4: Below is the graph of one cycle of a tangent (cotangent) function, $y = f(t)$.



1. Write $f(t)$ in the form $J(t) = A\tan(\omega t + \phi) + B$ for $\omega > 0$.

2. Write $f(t)$ in the form $K(t) = A\cot(\omega t + \phi) + B$ for $\omega > 0$.

EXAMPLE 5: Let θ be the angle of inclination from an observation point on the ground 42 feet away from the launch site of a model rocket. Assuming the rocket is launched directly upwards:

1. Find a formula for $f(\theta)$, the distance from the rocket to the ground (in feet).

   Find and interpret $f\left(\frac{\pi}{3}\right)$.

2. Find a formula for $g(\theta)$, the distance from the rocket to the observation point on the ground (in feet).

   Find and interpret $g\left(\frac{\pi}{3}\right)$.

3. Find and interpret the behavior of $f(\theta)$ and $g(\theta)$ as $\theta \to \frac{\pi}{2}^{-}$.

HOMEWORK: Section 10.5: Page 898: 1 - 16 all